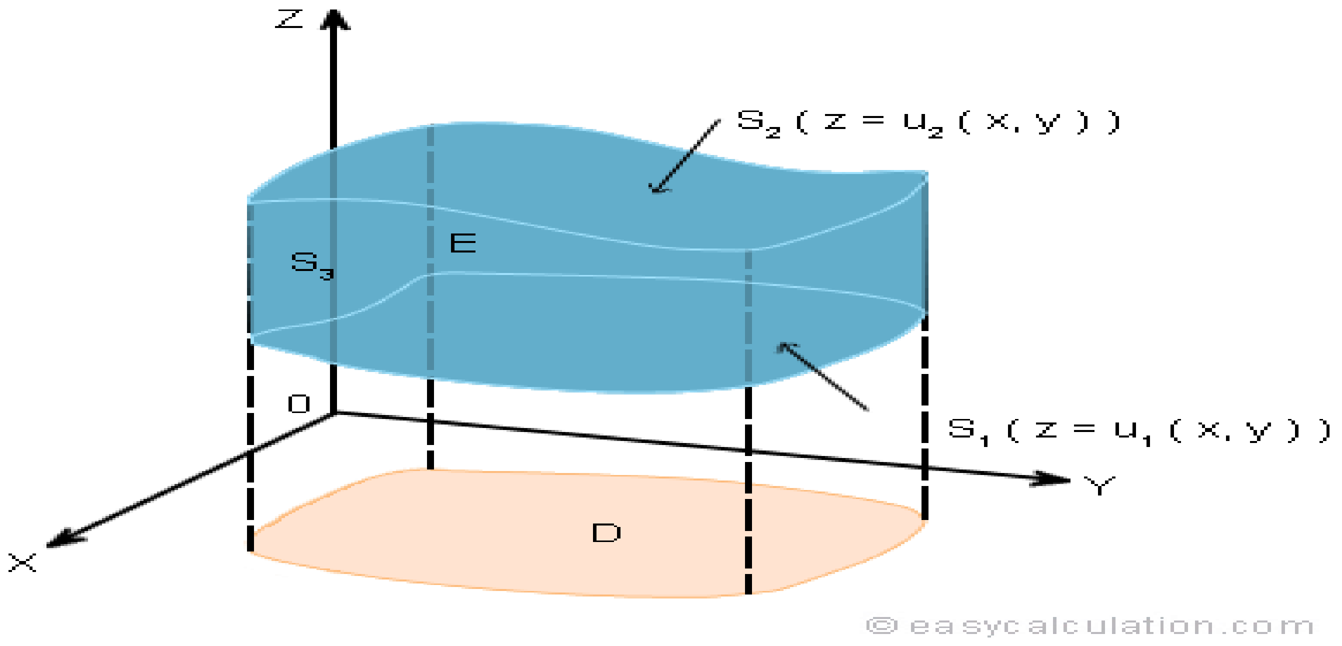
MTH 166

Lecture-34

Gauss's Divergence Theorem

Statement: Let D be a closed and bounded region in 3-dimensional space whose boundary is a piecewise smooth surface S oriented outwards. Let V be a vector which is continuous and have continuous first order partial derivatives. Let n is a unit normal vector drawn outward to the surface S. Then:

$$\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv = \iiint_D (\vec{\nabla}.\vec{V})dv$$



Important Results from MCQ point of view:

1. By gauss divergence theorem: $\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv = \iiint_D (\vec{\nabla}.\vec{V})dv$
2. Gauss Divergence theorem gives a relationship between double integral and triple integral, unlike Green's theorem and Stokes' theorem.
3. Gauss Divergence theorem is applicable to a closed region bounded by a surface, where as Stokes' theorem is applicable to an open surface bounded by a closed curve.
4. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}, \vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$(a constant vector) and $V$is the volume,

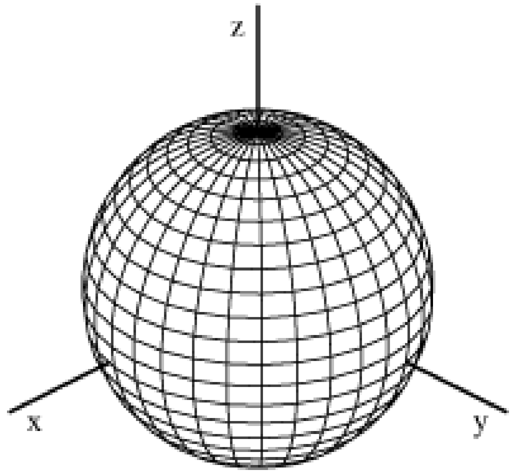
(I). $\iint_S (\vec{a}.\hat{n})dA = 0$

(II). $\iint_S (curl\,\vec{r}.\hat{n})dA = 0$

(III). $\iint_S (\vec{r}.\hat{n})dA = 3V = 3(Volume\ of\ the\ given\ region)$

Problem 1: Evaluate $\iint_S \left(\vec{V}.\hat{n}\right)dA$ using Gauss Divergence Theorem,

where $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and D is the region bounded by the sphere $x^2 + y^2 + z^2 = 16$.



Solution: Here: $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\Rightarrow div\vec{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$$

$$\Rightarrow div\vec{V} = 3$$

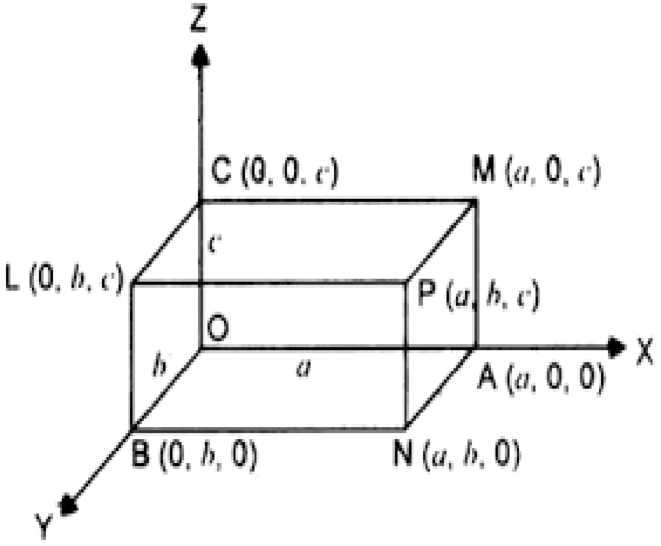
By Gauss Divergence Theorem: $\iint_S \left(\vec{V}.\hat{n}\right)dA = \iiint_D \left(div\,\vec{V}\right)dv$

$$\Rightarrow \iint_S \left(\vec{V}.\hat{n}\right)dA = \iiint_D 3dv = 3\iiint_D dv = 3(Volume\ of\ sphere x^2 + y^2 + z^2 = 16.)$$

$$\Rightarrow \iint_S \left(\vec{V}.\hat{n}\right)dA = 3\left(\frac{4}{3}\pi r^3\right) = 4\pi(4^3) = 256\pi$$ **Answer.**

Problem 2: Evaluate $\iint_S (\vec{V}.\hat{n})dA$ using Gauss Divergence Theorem,

where $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and D is bounded by the edges $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.



Solution: Here: $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\Rightarrow div\vec{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$$

$$\Rightarrow div\vec{V} = 3$$

By Gauss Divergence Theorem: $\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv$

$$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \iiint_D 3dv = 3\iiint_D dv = 3(Volume\ of$$ cuboid.)

$$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = 3(lbh) = 3(abc)$$ <u>**Answer.**</u>

Problem 3: Evaluate $\iint_S \left(\vec{V}.\hat{n}\right)dA$ using Gauss Divergence Theorem,

where $\vec{V} = 3x^2\hat{\imath} + 6y^2\hat{\jmath} + z\hat{k}$ and D is the region bounded by the closed cylinder $x^2 + y^2 = 16$, z = 0 and z = 4.

Solution: Here: $\vec{V} = 3x^2\hat{\imath} + 6y^2\hat{\jmath} + z\hat{k}$

$$\Rightarrow div\vec{V} = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(6y^2) + \frac{\partial}{\partial z}(z)$$

$$\Rightarrow div\vec{V} = 6x + 12y + 1$$

By Gauss Divergence Theorem: $\iint_S \left(\vec{V}.\hat{n}\right)dA = \iiint_D \left(div\,\vec{V}\right)dv$

$$\Rightarrow \iint_S \left(\vec{V}.\hat{n}\right)dA = \iiint_D (6x + 12y + 1)dv$$

Let us calculate the limits of x, y, z

$0 \le z \le 4$ (Given height of cylinder)

Also $x^2 + y^2 = 16$

$\Rightarrow y = \pm\sqrt{16 - x^2}$

$\Rightarrow -\sqrt{16 - x^2} \le y \le \sqrt{16 - x^2}$

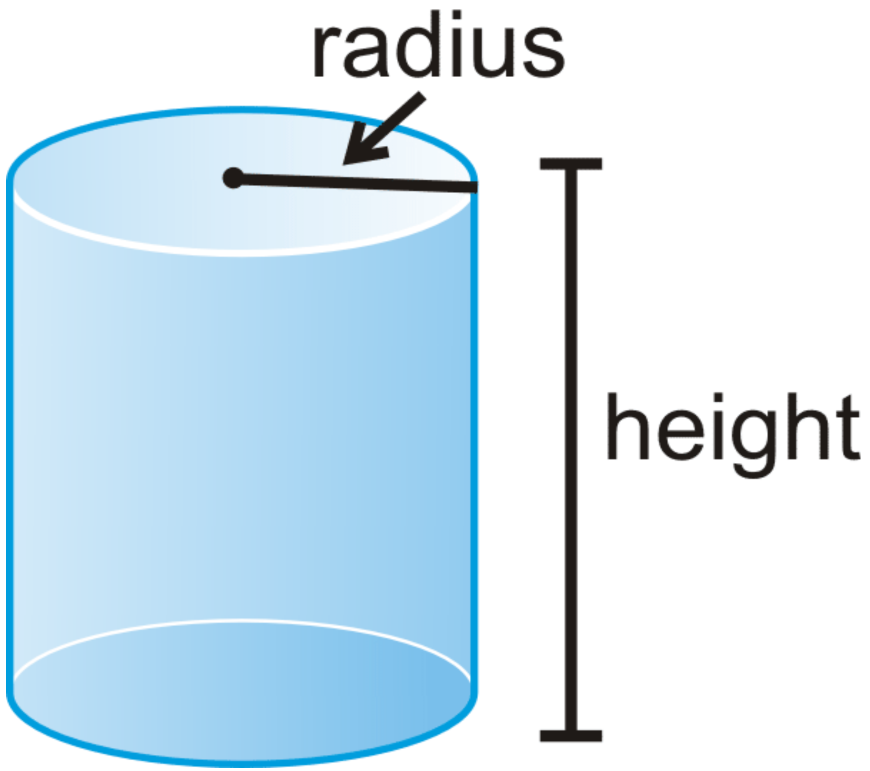
On x-axis (y=0)

$\Rightarrow x^2 + 0^2 = 16$

$\Rightarrow x = \pm 4$

$\Rightarrow -4 \le x \le 4$

Let us put these limits in the given integral



$$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \iiint_D (6x + 12y + 1)dv$$

$$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \int_{z=0}^{z=4}\int_{x=-4}^{x=4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}}(6x + 12y + 1)dydxdz$$

$$= \int_{z=0}^{z=4} dz \int_{x=-4}^{x=4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}}(6x + 12y + 1)dydx$$

$$= 4\int_{x=-4}^{x=4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}}(6x + 12y + 1)dydx$$

$$= 4\int_{x=-4}^{x=4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}}(1)dydx$$ (Because x and y are odd functions)

$$= 4(2)(2)\int_{x=0}^{x=4}\int_{y=0}^{\sqrt{16-x^2}}(1)dydx$$ (Because 1 is an even function)

$$= 16\int_{x=0}^{4}\sqrt{16 - x^2}dx = 16\left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2}sin^{-1}\left(\frac{x}{4}\right)\right]_{x=0}^{4}$$

$= 64\pi$ **Answer.**